

## Mathematics

- Q.1 A line passes through the point  $P(h, k)$  is parallel to the  $x$ -axis. It forms a triangle with the lines  $y = x$  &  $x + y = 2$  of area  $4h^2$  then find the locus of  $P$ . [2]

Sol. On solving the lines we get  $S$  &  $R$  points

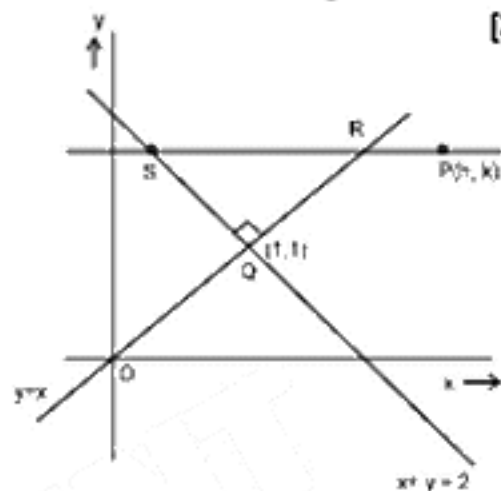
$$S : (2 - k, k), R : (k, k)$$

Two lines intersect at  $Q(1, 1)$

$$\text{Area of the } (\Delta QRS) = \frac{1}{2} \times 2(k-1)^2 \quad [\because QS = QR = \sqrt{2} SR]$$

$$4h^2 = [k-1]^2$$

$$\Rightarrow \text{locus is the pair of straight lines} \quad \boxed{4x^2 = (y-1)^2} \quad \text{Ans.}$$



- Q.2 A cricket player in his career plays  $n$  match and scores total no. of  $\frac{(n+1)(2^{n+1}-n-2)}{4}$  runs. [2]

Sol. If he scores  $k \cdot 2^{n-k+1}$  runs in  $k^{\text{th}}$  match, where  $1 \leq k \leq n$ . Find  $n$ .

Let  $S_n$  be the total scores in his career plays  $n$  matches.

$$S_n = \sum_{k=1}^n k \cdot 2^{n-k+1} = 2^{n+1} \sum_{k=1}^n k \cdot 2^{-k}$$

$$= 2^{n+1} \left[ \frac{\frac{1}{2} \left( 1 - \frac{1}{2^n} \right)}{1 - \frac{1}{2}} - \frac{n}{2^{n+1}} \right] \frac{1}{1 - \frac{1}{2}}$$

$$= 2^{n+2} \left[ 1 - \frac{1}{2^n} - \frac{n}{2^{n+1}} \right] = 2^{n+2} - 4 - 2n = 2^{n+2} - 2n - 4$$

$$\text{But } S_n = \frac{n+1}{4} (2^{n+1} - 2 - n) \text{ (as given)}$$

$$\text{so } \frac{n+1}{4} (2^{n+1} - 2 - n) = 2(2^{n+1} - n - 2)$$

$$n+1 = 8 \quad \text{so } \boxed{n=7} \text{ Ans.}$$

- Q.3 Ramesh goes to office either by car, scooter, bus or train probability of which being  $\frac{1}{7}$ ,  $\frac{3}{7}$ ,  $\frac{2}{7}$  and  $\frac{1}{7}$  respectively and

probability that he is reaching office late if he takes car, scooter, bus or train is  $\frac{2}{9}$ ,  $\frac{1}{9}$ ,  $\frac{4}{9}$  and  $\frac{1}{9}$  respectively. Find the

probability that he has travelled by car, if he reaches office in time. [2]

Sol. Let  $A$ ,  $B$ ,  $C$ ,  $D$  be the events when Ramesh is going by car, scooter, bus or train respectively.

$$P(A) = \frac{1}{7}, P(B) = \frac{3}{7}, P(C) = \frac{2}{7}, P(D) = \frac{1}{7}$$

Let  $\bar{x}$  be the event when Ramesh reaching the office in time.

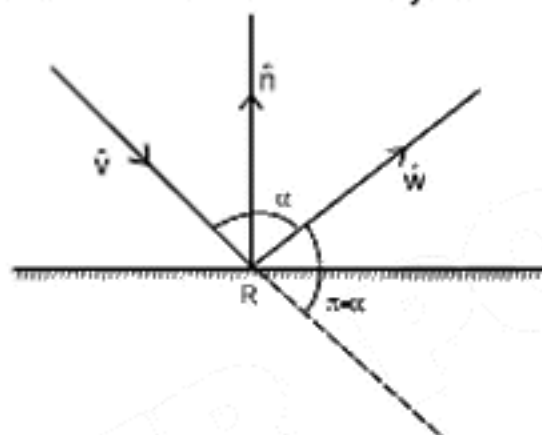
$$P(\bar{x}) = \frac{7}{9}, P(\bar{x}|B) = \frac{8}{9}, P(\bar{x}|C) = \frac{5}{9}, P(\bar{x}|D) = \frac{8}{9}$$

$$P\left(\frac{A}{\bar{x}}\right) = \frac{P(\bar{x}|A)P(A)}{P(\bar{x})} = \frac{\frac{7}{9} \times \frac{1}{7}}{\frac{1}{7} \times \frac{7}{9} + \frac{3}{7} \times \frac{8}{9} + \frac{2}{7} \times \frac{5}{9} + \frac{8}{9} \times \frac{1}{7}} = \frac{1}{7} \text{ Ans.}$$

Q.4 A light ray incident along the unit vector  $\hat{v}$  and the unit vector along the normal to reflecting surface at the point P is  $\hat{n}$  outwards. If reflecting ray is along the unit vector  $\hat{w}$ , find  $\hat{w}$  in terms of  $\hat{v}$  and  $\hat{n}$ . [2]

Sol.  $\hat{n}$  is bisector of the angle between the incident and reflected ray and can be represented as

$$\hat{n} = \frac{\hat{w} - \hat{v}}{|\hat{w} - \hat{v}|}$$



where  $|\hat{w} - \hat{v}| = \sqrt{1 + 1 - 2(\hat{w} \cdot \hat{v})}$  [where  $\pi - \alpha$  is angle between  $\hat{w}$  and  $\hat{v}$ ]

$$= |2 \cos \alpha/2| \quad \text{where } \alpha \in (0, \pi)$$

$$= -2(\hat{n} \cdot \hat{v})$$

$$\Rightarrow \hat{w} = \hat{v} - 2(\hat{n} \cdot \hat{v}) \hat{n} \quad \text{Ans.}$$

Q.5 Find the equation of the plane at distance of  $\frac{1}{\sqrt{6}}$  from the point  $(2, 1, -1)$  and containing the line [2]

$$2x - y + z - 3 = 0 = 3x + y + z - 5.$$

Sol.  $(2x - y + z - 3) + \lambda(3x + y + z - 5) = 0$

$$(3\lambda + 2)x + (\lambda - 1)y + (\lambda + 1)z - 5\lambda - 3 = 0$$

$$\frac{|8\lambda + 4 + \lambda - 1 - \lambda - 1 - 5\lambda - 3|}{\sqrt{(3\lambda + 2)^2 + (\lambda - 1)^2 + (\lambda + 1)^2}} = \frac{1}{\sqrt{6}}$$

$$6(\lambda - 1)^2 = 11\lambda^2 + 12\lambda + 6$$

$$\lambda = 0, -\frac{24}{5}$$

So the planes are  $2x - y + z - 3 = 0$  and  $62x + 29y + 19z - 105 = 0$  Ans.

**Q.6** For any two distinct real numbers  $x_1$  &  $x_2$ ,  $y = f(x)$  is satisfying the condition,  $|f(x_1) - f(x_2)| < (x_1 - x_2)^2$ . Find the equation of the tangent at the point (1, 2) to the curve  $y = f(x)$ . [2]

**Sol.** Given :  $|f(x_1) - f(x_2)| < |x_1 - x_2|^2$   
 Let  $x_2 = 1$  &  $x_1 = 1 + h$   
 $|f(1+h) - f(1)| < |h|^2$

$$\text{so } \lim_{h \rightarrow 0} \left| \frac{f(1+h) - f(1)}{h} \right| < \lim_{h \rightarrow 0} |h|$$

$|f'(1)| < \epsilon$ ; where  $\epsilon$  is a very small positive quantity.  
 so  $f'(1) = 0$   
 Hence the tangent to  $y = f(x)$  at (1, 2) is  $y = 2$ . **Ans.**

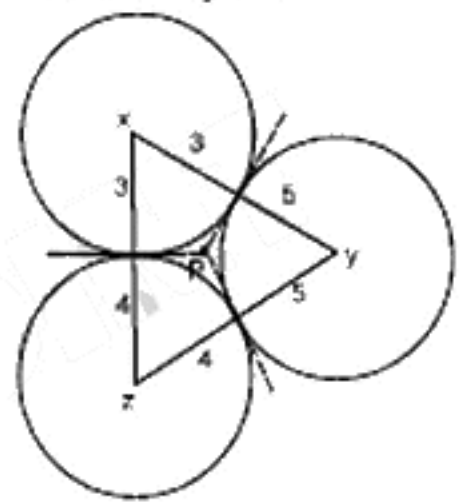
**Q.7** 3, 4, 5 are radii of three circles touch each other externally if P is the point of intersection of tangents of these circles at their points of contact, find the distances of P from the points of contact. [2]

**Sol.** Let x, y, z be the centre of the three circles clearly the point P is the in-centre of the  $\Delta xyz$  and

$$r = \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s}; \text{ where } \Delta \text{ \& \ } S \text{ have their usual meaning}$$

$$2s = 7 + 8 + 9 \Rightarrow s = 12$$

$$r = \sqrt{\frac{5 \cdot 4 \cdot 3}{12}} = \sqrt{5} \text{ unit. Ans.}$$



**Q.8** Evaluate :  $\int_0^{\frac{\pi}{2}} e^{\cos x} [2 \sin(1/2 \cos x) - 3 \cos(1/2 \cos x)] \sin x \, dx$  [2]

**Sol.** Let  $\frac{1}{2} \cos x = t$

$$\cos x = 2t \text{ so } -\sin x \, dx = 2dt$$

$$\text{so } I = \int_{-1/2}^{1/2} e^{2t} [2 \sin(t) + 3 \cos(t)] 2dt$$

$$= 2 \int_{-1/2}^{1/2} 2 \cdot e^{2t} \sin t \, dt + 6 \int_{-1/2}^{1/2} e^{2t} \cos t \, dt \quad \left( \text{Using } \int_{-a}^a f(x) \, dx = 0; \text{ if } f(x) \text{ is odd function} \right)$$

$$= 2 \int_0^a f(x) \, dx; \text{ if } f(x) \text{ is even function}$$

$$= 0 + 12 \int_0^{1/2} e^{2t} \cos t \, dt$$

$$= \frac{12}{\sqrt{5}} [e^{2t} \cos(t - \tan^{-1} \frac{1}{2})]_0^{1/2}$$

$$= \frac{12}{\sqrt{5}} [e \cos(\frac{1}{2} - \tan^{-1} \frac{1}{2}) - \cos(\tan^{-1} \frac{1}{2})]$$

$$= \frac{12}{\sqrt{5}} [e \cos(\frac{1}{2} - \tan^{-1} \frac{1}{2}) - \frac{2}{\sqrt{5}}] \text{ Ans.}$$

**Q.9** Find the area bounded by the curves  $x^2 = y$ ,  $x^2 = -y$  &  $y^2 = 4x - 3$ . [4]

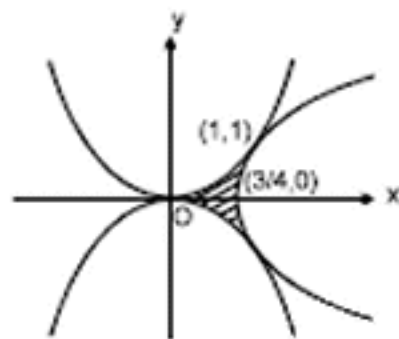
**Sol.** On solving the curves  $x^2 = y$  and  $y^2 = 4x - 3$

We get their point of contact  $(1, 1)$  and the area bounded is symmetric about x-axis.

so that the area bounded is

$$= 2 \left[ \int_0^{3/4} x^2 + \int_{3/4}^1 (x^2 - \sqrt{4x-3}) dx \right]$$

$$= \frac{1}{3} \text{ sq. units. Ans.}$$



**Q.10** Find the equation of the common tangent in 1st quadrant to the circle  $x^2 + y^2 = 16$  and the ellipse  $\frac{x^2}{25} + \frac{y^2}{4} = 1$ . Also find [4]

the length of the intercept of the tangent between the coordinates axes.

**Sol.** Let the equations of tangents to the given circle and the ellipse respectively.

$$y = mx + 4\sqrt{1+m^2} \quad \dots(1)$$

$$\text{and } y = mx + \sqrt{25m^2 + 4} \quad \dots(2)$$

since (1) & (2) are coincident lines, so

$$4\sqrt{1+m^2} = \sqrt{25m^2 + 4}$$

$$16(1+m^2) = 25m^2 + 4$$

$$m = \pm \frac{2}{\sqrt{3}}$$

$m < 0$  because common tangent in 1st quadrant

$$\text{so } m = -\frac{2}{\sqrt{3}}$$

so the equation of the common tangent is ; (from (1))

$$y = -\frac{2}{\sqrt{3}}x + 4\sqrt{\frac{7}{3}}$$

It meets the coordinates axes at A  $(2\sqrt{7}, 0)$  and  $(0, 4\sqrt{\frac{7}{3}})$  so length of the intercept of the tangent between the

coordinate axes is  $\frac{14}{\sqrt{3}}$ . **Ans.**

**Q.11** A tangent is drawn from a point on the hyperbola  $\frac{x^2}{9} - \frac{y^2}{4} = 1$  to circle  $x^2 + y^2 = 9$  find the locus of mid-point of chord of contact. [4]

**Sol.** Let any point  $P(3 \sec\theta, 2 \tan\theta)$  on the hyperbola  
chord of contact of the circle  $x^2 + y^2 = 9$  with respect to the point  $(3 \sec\theta, 2 \tan\theta)$  is  
 $(3 \sec\theta) \cdot x + (2 \tan\theta) y = 9$  .....(1)

Let  $(h, k)$  be the mid point of the chord of contact.

$$\text{equation of chord in mid-point form is } xh + yk = h^2 + k^2 \quad \dots(2)$$

By (1) and (2)

$$\frac{3 \sec \theta}{h} = \frac{2 \tan \theta}{k} = \frac{9}{h^2 + k^2}$$

$$\sec \theta = \frac{3(h^2 + k^2)}{h}, \quad \tan \theta = \frac{9k}{2(h^2 + k^2)}$$

as  $\sec^2 \theta - \tan^2 \theta = 1$ , so

$$\frac{81h^2}{9(h^2 + k^2)^2} - \frac{81k^2}{4(h^2 + k^2)^2} = 1$$

so the required locus is  $(x^2 + y^2)^2 = 9x^2 - \frac{81}{4}y^2$  Ans.

**Q.12** A square having one vertex as  $2 + \sqrt{3}i$  is circumscribed on the circle  $|z - 1| = \sqrt{2}$ . Then find the other vertices of square. [4]

**Sol.** Centre of circle is  $(1, 0)$  and is also the mid point of diagonals of square

$$z_3 - z_0 = (z_1 - z_0) e^{i\pi/2} \\ = (2 + \sqrt{3}i - 1)i$$

$$z_3 = i - \sqrt{3} + 1$$

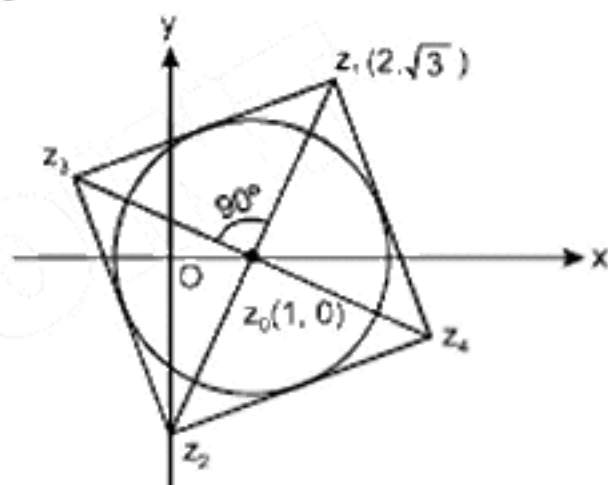
$$= (1 - \sqrt{3} + i)$$

Since  $z_0$  is mid point of  $z_1$  &  $z_2$  as well as  $z_3$  &  $z_4$  also

$$\text{so } z_4 = 2z_0 - z_3 = (1 + \sqrt{3}) - i$$

$$\text{and } z_2 = 2z_0 - z_1 = 2 - (2 + \sqrt{3}i) = -\sqrt{3}i$$

so other vertices are  $(0, -\sqrt{3}), (1 - \sqrt{3}, 1), (1 + \sqrt{3}, -1)$  Ans.



**Q.13** Find all the curves  $y = f(x)$ , tangents at any point on which are drawn such that the segment of tangent intercepted between the contact point and x-axis is of unit length. [4]

**Sol.** Given :  $|AP| = 1$

Equation of tangent at  $P(x, y)$  point is

$$Y - y = \frac{dy}{dx} (X - x)$$

Putting  $X = 0$  &  $Y = 0$  respectively we get pts A & B and length of the segment AB is

$|AP|$  is length of tangent, so

$$|AP| = \left| \frac{y}{y'} \sqrt{1 + (y')^2} \right| = 1 \quad \text{where } y' = \left( \frac{dy}{dx} \right)_P$$

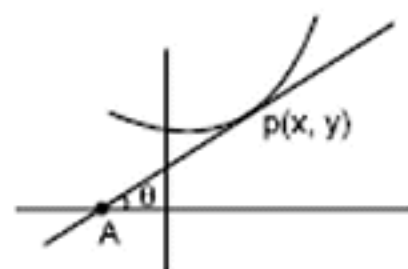
$$y^2 \left( 1 + \left( \frac{dy}{dx} \right)^2 \right) = \left( \frac{dy}{dx} \right)^2$$

$$\int \frac{\sqrt{1 - y^2}}{y} dy = \pm \int dx$$

on integrating we get,

$$\sqrt{1 - y^2} + \ln \left| \frac{1 - \sqrt{1 - y^2}}{y} \right| = \pm x + c$$

Ans.



- Q.14 If two functions 'f' and 'g' satisfying given conditions for  $\forall x, y \in \mathbb{R}$ .  $f(x - y) = f(x)g(y) - f(y) \cdot g(x)$  and  $g(x - y) = g(x) \cdot g(y) + f(x) \cdot f(y)$ . If right hand deviative at  $x = 0$  exists for  $f(x)$  then find the deviative of  $g(x)$  at  $x = 0$ . [4]

Sol. Given  $f(x - y) = f(x) \cdot g(y) - f(y) \cdot g(x)$   
 Put  $y = x$  and you get  $f(0) = 0$   
 put  $y = 0$  and you get  $g(0) = 1$

$$\begin{aligned} \text{R.H.D. of } f(x): f'(0^+) &= \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} \quad (h \in \mathbb{R}^+ \text{ and tending to zero}) \\ &= \lim_{h \rightarrow 0} \frac{f(0)g(-h) - g(0)f(-h) - f(0)}{h} = \lim_{h \rightarrow 0} \frac{-f(-h)}{h} = f'(0^-) = \text{L.H.D.} \quad \dots(1) \end{aligned}$$

$$\text{\& L.H.D. of } f(x): f'(0^-) = \lim_{h \rightarrow 0} \frac{f(0-h) - f(0)}{-h} = -f'(0^+) = -\text{R.H.D.} \quad \dots(2)$$

Hence from (1) and (2)  $f'(0) = 0$

Put  $y = x$  in given condition  $g(x - y) = g(x) \cdot g(y) + f(x) \cdot f(y)$

$$\Rightarrow g(0) = g^2(x) + f^2(x)$$

on diff. w.r.t.x we get  $\Rightarrow g^2(x) + f^2(x) = 1 \Rightarrow g^2(x) = 1 - f^2(x)$

$$\Rightarrow 2g(x) \cdot g'(x) + 2f(x) f'(x) = 0$$

$$g'(0) = 0$$

[Note : 'g' is differentiable at zero because 'f' is diff. at 0 &  $g^2(x) = 1 - f^2(x)$ ]

- Q.15 Find the range of values of t for which  $2 \sin t = \frac{1-2x+5x^2}{3x^2-2x-1} ; t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$  [4]

Sol. Given:  $2 \sin t = \frac{1-2x+5x^2}{3x^2-2x-1}$

First we have to check the range of R.H.S.  $\frac{1-2x+5x^2}{3x^2-2x-1} = y$  (say)

$$(3x^2 - 2x - 1)y - (1 - 2x + 5x^2) = 0$$

$$(3y - 5)x^2 - 2x(y - 1) - (y + 1) = 0$$

So  $D \geq 0$

since  $x \in \mathbb{R} - \{1, -\frac{1}{3}\}$

$$y^2 - y - 1 \geq 0$$

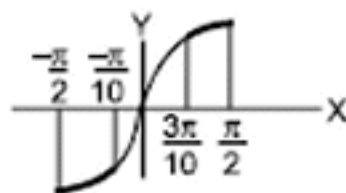
$$\text{or } y \geq \frac{1+\sqrt{5}}{2} \quad \text{or} \quad y \leq \frac{1-\sqrt{5}}{2}$$

$$(\because \sin \frac{\pi}{10} = \frac{\sqrt{5}-1}{4} \text{ \& } \sin \frac{3\pi}{10} = \frac{\sqrt{5}+1}{4})$$

$$\text{or } \sin t \geq \frac{1+\sqrt{5}}{4} \quad \text{or} \quad \sin t \leq \frac{1-\sqrt{5}}{4}$$

$$\Rightarrow \sin t \geq \sin \frac{3\pi}{10} \quad \text{or} \quad \sin t \leq \sin \left(-\frac{\pi}{10}\right)$$

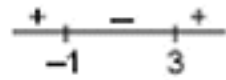
$$\Rightarrow \boxed{\frac{\pi}{2} \geq t \geq \frac{3\pi}{10} \quad \text{or} \quad -\frac{\pi}{2} \leq t \leq -\frac{\pi}{10}} \quad \text{Ans.}$$



- Q.16 If  $P(x)$  be the cubic polynomial satisfying  $p(-1) = 10$ ,  $p(1) = -6$  and  $p(x)$  has maximum at  $x = -1$  and  $p'(x)$  has minima at  $x = 1$ . Find the points of local maxima and minima, also find the distance between these two points. [4]

Sol. Let  $P(x) = ax^2 + bx^2 + cx + d$   
 $P(-1) = -a + b - c + d = 10$  ....(1)  
 &  $P(1) = a + b + c + d = -6$  ....(2)  
 $P'(x) = 3ax^2 + 2bx + c$   
 since  $P'(-1) = 3a - 2b + c = 0$  ....(3)  
 and  $P''(x) = 6ax + 2b$   
 $P''(1) = 6a + 2b = 0$

solving (1), (2), (3) & (4) for a, b, c, d  
 $a = 1, b = -3, c = -9, d = 5$   
 so  $P(x) = x^3 - 3x^2 - 9x + 5$   
 Now  $P'(x) = 3x^2 - 6x - 9 = 3(x^2 - 2x - 3) = 3(x - 3)(x + 1)$



So  $P(x)$  has minima at  $x = 3$  & maxima at  $x = -1$   
 so the point of local max.  $(-1, 10)$   
 so the point minima  $(3, -22)$

distance =  $\sqrt{16 + (32)^2} = \sqrt{16 + 1024} = \sqrt{1040} = 4\sqrt{65}$  unit Ans.

Q.17  $f(x)$  be a quadratic polynomial, a, b, c are three distinct real numbers, such that :

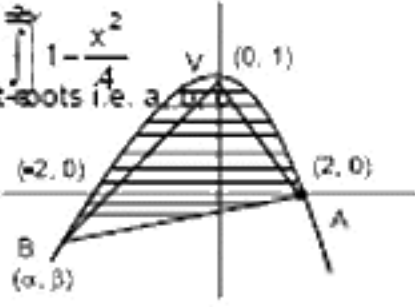
$$\begin{bmatrix} 4a^2 & 4a & 1 \\ 4b^2 & 4b & 1 \\ 4c^2 & 4c & 1 \end{bmatrix} \begin{bmatrix} f(-1) \\ f(1) \\ f(2) \end{bmatrix} = \begin{bmatrix} 3a^2 + 3a \\ 3b^2 + 3b \\ 3c^2 + 3c \end{bmatrix}$$

V is the point where  $f(x)$  attains maximum. A & B are the points on  $f(x)$  such that  $f(x)$  cuts x-axis at A in the first quadrant and chord AB subtends right angle at V. Find the area bounded by curve  $y = f(x)$  and chord AB. [6]

Sol. As given  $4a^2 f(-1) + 4a f(1) + f(2) = 3a^2 + 3a$   
 $4b^2 f(-1) + 4b f(1) + f(2) = 3b^2 + 3b$  and  
 $4c^2 f(-1) + 4c f(1) + f(2) = 3c^2 + 3c$

here it can be considered that the equation:  
 $4x^2 f(-1) + 4x f(1) + f(2) = 3x^2 + 3x$  has three distinct roots i.e. a, b, c  
 which is possible if and only if

$4f(-1) = 3 \Rightarrow f(-1) = 3/4$   
 and  $4f(1) = 3 \Rightarrow f(1) = 3/4$   
 $f(2) = 0$



so the function  $f(x) = \frac{-x^2}{4} + 1$

Let A be at  $(2, 0)$ . let B be  $(\alpha, \beta)$

so  $\beta = -\frac{\alpha^2}{4} + 1$  & .....(1)

Since  $BV \perp AV$  so

$-\frac{1}{2} \left( \frac{\beta - 1}{\alpha} \right) = -1 \Rightarrow \beta = 2\alpha + 1$  .....(2)

from (1) & (2)  
 $(\alpha, \beta) = (-8, -15) =$  point B

so the required area =  $\int_{-8}^2 \left( 1 - \frac{x^2}{4} \right) dx$

area =  $\frac{125}{3}$  sq. units

 Ans.

Q.18 If 'f' is differentiable and 'g' is a double differentiable function such that  $f(x) \in [-1, 1]$  and  $f'(x) = g(x)$ .

If  $f^2(0) + g^2(0) = 9$  then prove that there exist some  $c \in (-3, 3)$  such that  $g(c) \cdot g''(c) < 0$  [6]

Sol. We have to prove that  $g(x) \cdot g''(x) < 0$  for some  $c \in (-3, 3)$  means both can not be +ve or -ve for all  $x \in (-3, 3)$  simultaneously. First we assume both are +ve for all  $x \in (-3, 3)$ .

$f(x) \in [-1, 1]$  and  $f^2(0) = 9$

$$\Rightarrow g(0) \in [2\sqrt{2}, 3]$$

given  $f'(x) = g(x)$

$$\int_{-3}^x f'(x) dx = \int_{-3}^x g(x) dx$$

$$f(x) = \int_{-3}^x g(x) dx + f(-3) \quad [f(-3) \text{ can take min}^m \text{ value } -1]$$

since  $g''(x) > 0 \Rightarrow$  curve is opening upwards

for any  $g(x)$  satisfying the conditions, why?

you can understand by following cases -

Case I : If  $g(x)$  is decreasing.

$$\int_{-3}^x g(x) dx > 3 \times 2\sqrt{2} \quad (\text{area of rectangle})$$

Case II : If  $g(x)$  is increasing

$$\int_{-3}^x g(x) dx > 6\sqrt{2} \quad (\text{area of rectangle})$$

Case III : If  $g(x)$  takes min<sup>m</sup> at  $x = 0$

$$\int_{-3}^x g(x) dx > 6 \times 2\sqrt{2} > 6\sqrt{2} \quad (\text{area of rectangle})$$

$$f(x) > 6\sqrt{2} - 1$$

but  $f(x)$  can not be greater than one so that their shows contradiction means assumed condition can not be true any how.

$\Rightarrow g(x)$  and  $g''(x)$  can not be both +ve for simultaneously all  $x \in (-3, 3)$

for some  $c \in (-3, 3)$   $g(c) \cdot g''(c) < 0$

similarly you can prove that both can not be -ve simultaneously.

